#9: Linear Quadratic Optimum Control

Given a controllable system in state-space form, you can always stabilize the system using full-state feedback. So far, the only method we have to find these feedback gains is pole placement. The power of pole placement is that you can place the system poles anywhere. The designer is likewise free to shape the system's response any way he/she desires.

The strength of pole placement is also its weakness. Since the system's poles can be placed anywhere, the designer is not given any help is choosing where the poles *should* be placed. This is a problem since, when practical considerations are taken into account, such as noise effects, limited controller size, finite word length, etc. some pole locations are better than others.

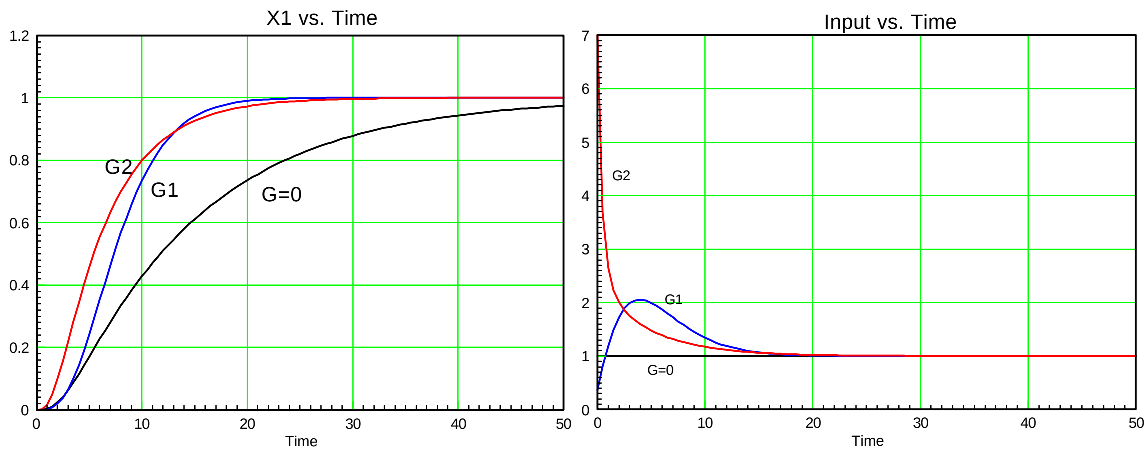
For example, consider the problem of controlling the tip temperature of a metal bar:

$$sX = \begin{bmatrix} -a & a & 0 & 0 \\ a & -2a & a & 0 \\ 0 & a & -2a & a \\ 0 & 0 & a & -2a \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 0 \\ a \end{bmatrix} U \qquad a=0.6414$$

The uncontrolled system has a settling time of about 40 seconds. If you want to double this rate, full-state feedback can be used to place the poles. The dominant pole should be around -0.2. The three remaining are arbitrary - at least in terms of determining the settling time. If you look at the feedback gains and the input, however, some pole locations are better than others. For example, two possible locations (along with the uncontrolled dynamics) are:

```
poles = [-0.08, -0.64, -1.50, -2.27]  G0 = [0.0000,  0.0000, 0.0000,  0.0000]
poles = [-0.50, -0.50, -0.50, -0.50]  G1 = [4.1758, -8.9799, 8.0552, -3.8813]
poles = [-0.20, -1.00, -2.00, -3.00]  G2 = [0.9930,  1.1075, 1.3235,  2.6664]
```

The step response for these three follow:



Note that

- The output (X1) is about the same for both sets of gains, G1 and G2.
- The gains associated with G2 appear better since they are smaller and all positive (so you don't add more heat as a temperature increases as is the case with two gains in G1).
- The input to the system for G1 is better than for G2 since it responds less rapidly (making it less susceptible to noise) and is smaller (requiring a smaller actuator to power this system).

In short, some gains are better than others. Ideally, you would like to find a set of feedback gains so that

- The output responds as quickly as possible
- Without excessive overshoot
- Without requiring a large input

• With the input and output responding "smoothly" so that the system isn't sensitive to noise.

This chapter presents a way to choose the "optimal" location to place the closed-loop poles.

9.2 Formulation of the Optimal Control Problem: Calculus of Variations

Calculus of Variations is a branch of mathematics dealing with optimizing functionals. A functional is a function of functions. For example,

$$J(x) = \int_a^b F(t, x, x')dt$$

computes a cost, J, for a given function x(t). For different x(t)'s, you'll have different costs. The problem of finding x(t) which minimizes (or maximizes) J is generally the problem solved with calculus of variations.

Euler Legrange Equation with One Dependent Variable:

The minimum is found from

$$\frac{dJ}{d\xi} = \lim_{\xi \to 0} \left(\frac{J(x+\xi n) - J(x)}{\xi} \right) = 0$$

Expanding this using a Taylor's series (where h.o.t. means higher order terms)

$$\begin{aligned} F(t, x+\xi n, x'+\xi n') &= F(t, x, x') + \xi \tfrac{dF}{d\xi} + h.o.t. \\ &= F(t, x, x') + \xi\left(\tfrac{\partial F}{\partial x} n + \tfrac{\partial F}{\partial x'} n' \right) + h.o.t. \end{aligned}$$

The partial is then

$$\begin{aligned} \frac{dJ}{d\xi} &= \lim_{\xi \to 0} \left(\frac{\int_a^b (F(t, x+\xi n, x'+\xi n') - F(t, x, x'))dx}{\xi} \right) \\ &= \lim_{\xi \to 0} \left(\frac{\xi\left(\int_a^b \left(F_x n + F_{x'} n' \right) dt \right) + h.o.t.}{\xi} \right) \end{aligned}$$

Taking the limit results in

$$\frac{dJ}{d\xi} = \int_a^b (F_x n + F_{x'} n')dt$$

Note that

$$F_{x'} n' = \tfrac{d}{dx}(F_{x'} n) - \tfrac{d}{dx}(F_{x'})n$$

allowing you to write this integral as

$$\frac{dJ}{d\xi} = \int_a^b \left(\left(F_x - \tfrac{d}{dt}(F_{x'}) \right) n + \tfrac{d}{dt}(F_{x'} n) \right) dt = 0$$

or

$$\int_a^b \left(F_x - \tfrac{d}{dt}(F_{x'}) \right) n \cdot dt + \left| (F_{x'} n) \right|_a^b = 0$$

Since n(t) is an arbitrary function, this can only be true if

$$\boxed{F_x - \tfrac{d}{dt}(F_{x'}) = 0}$$ **Euler Legrange Equation**

If a function x(t) minimizes a cost function, J, then x(t) must satisfy the Euler Legrange equation.

The boundary conditions are then set by the second term.

- If x(a) is fixed, n(a)=0
- If x(a) is free, then $F_{x'} = 0$.

Example: Find x(t) which minimizes the functional

$$J = \int_0^1 (x^2 + \dot{x}^2)dt$$

subject to the constraints that x(0)=1, x(1)=0.

Solution: The Euler Legrange equation gives

$$2x - \frac{d}{dt}(2\dot{x}) = 0$$

$$x - x'' = 0$$

This is a differential equation with poles at s = {+1, -1}

$$x(t) = ae^t + be^{-t}$$

Plugging in the boundary conditions:

$$x(0) = 1 = a + b$$

$$x(1) = 0 = 2.7183a + 0.3679b$$

$$a = -0.1565, b = 1.1565$$

so the function which minimizes this functional is

$$x(t) = -0.1565e^t + 1.1565e^{-t}$$

Euler Legrange Equation with Two Dependent Variables:

Next, if you have *two* dependent variables,

$$J = \int_a^b F(x, x', u, u', t)dt$$

you have *two* Euler Legrange equations to solve:

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial x'}\right) = 0$$

$$\frac{\partial F}{\partial u} - \frac{d}{dt}\left(\frac{\partial F}{\partial u'}\right) = 0$$

Euler Legrange Equation with Constraints:

Finally, if you have constraints, such as

$$G(x, x', u, u', t) = 0$$

you can modify the cost functional by adding a Legrange multiplier:

$$J = \int_a^b (F(x, x', u, u', t) + MG(x, x', u, u', t))dt$$

You can then solve this functional by plugging in the boundary conditions and the constraint on G(x,x',t).

Example 2: Find x(t) to minimize

$$J = \int_0^1 (x^2 + u^2)dt$$

subject to the constraints

$$\dot{x} = u, \text{ x(0)=1, x(1)=0}$$

Solution: F is now

$$F = (x^2 + u^2) + m(\dot{x} - u)$$

The Euler Legrange equations are

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial x'}\right) = 2x - \frac{d}{dt}(m) = 2x - m' = 0$$

$$\frac{\partial F}{\partial u} - \frac{d}{dt}\left(\frac{\partial F}{\partial u'}\right) = 2u - m = 0$$

so

$$m = 2u$$

$$2x = u'$$

and

$$x'' = u' = 2x$$

so

$$x(t) = ae^{\sqrt{2}\,t} + be^{-\sqrt{2}\,t}$$

Plugging in the initial conditions results in

$$x(0) = 1 = a + b$$

$$x(1) = 0 = 4.1133a + 0.2431b$$

$$a = -0.0628, b = 1.0628$$

and

$$x(t) = -0.0628e^{\sqrt{2}\,t} + 1.0628e^{-\sqrt{2}\,t}$$

$$u(t) = x'(t) = -0.0888e^{\sqrt{2}\,t} - 1.5030e^{-\sqrt{2}\,t}$$

Example 3: Find the functional to minimize

$$J = \int_a^b (X^TQX + U^TRU)dt$$

subject to the constraint

$$\dot{X} = AX + BU$$

Solution: The functional becomes

$$F = (X^TQX + U^TRU) + 2M^T\left(AX + BU - \dot{X}\right)$$

The Euler Legrange equations are then

$$2X^TQ + 2M^TA - \tfrac{d}{dt}(-2M^T) = 0$$

$$\dot{M}^T = -X^TQ - M^TA$$

$$\dot{M} = -QX - A^TM$$

and

$$2U^TR + 2M^TB = 0$$

$$RU = -B^TM \qquad \text{(R, M are symmetric)}$$

$$U = -R^{-1}B^TM$$

so you have a dynamic system

$$\begin{bmatrix} \dot{X} \\ \dot{M} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}\begin{bmatrix} X \\ M \end{bmatrix}$$

which can be solved subject to the constraints on X(a) and X(b).

Full-State Feedback Formulation:

Assume that

$$M = PX$$

so that the full-state feedback gains are

$$G = R^{-1}B^TP$$

Then the dynamics become

$$\dot{X} = (A - BR^{-1}B^TP)X$$

$$\dot{P}X + P\dot{X} = (-Q - A^TP)X$$

Multiplying the first equation by P and rearranging

$$P\dot{X} = (PA - PBR^{-1}B^TP)X$$

$$P\dot{X} = \left(-\dot{P} - Q - A^TP\right)X$$

This implies that

$$PA - PBR^{-1}B^TP = -\dot{P} - Q - A^TP$$

or

$$\boxed{\begin{aligned} \dot{P} &= -A^TP - PA - Q + PBR^{-1}BP \\ G &= -R^{-1}B^TP \end{aligned}}$$

Algebraic Ricatti equation for computing time-varying feedback gains: U=-GX

This gives the optimal time-varying feedback gain. If the feedback gains are to be constant, then $\dot{P} = 0$ and

$$\boxed{\begin{aligned} 0 &= A^TP + PA + Q - PBR^{-1}BP \\ G &= -R^{-1}B^TP \end{aligned}}$$

Algebraic Ricatti Equation you'll see most places:

$$0 = A^T M + MA + Q - MBR^{-1}BM$$
$$G = -R^{-1}B^T M$$

Algebraic Ricatti Equation you'll see in this text (U=-GX):

Example 3: For a first-order system:

$$\dot{x} = u$$

m is then

$$0 = -m^2/r + q$$

or $m = \sqrt{qr}$ and $g = \sqrt{q/r}$. Note that

- Only the ratio of q/r matters - not their absolute values. This is reasonable since U(t) which minimizes a functional is being found. Once found, this U(t) will also minimize 10 times that functional as well. Hence, you can scale Q and R at will and not change the results.
- As Q increases, the pole shifts left (more stable) as the square root of Q
- As R increases, the pole shifts right (slower response) as the square root of R^{-1}

Example 4:

$$\dot{x} = -x + u$$

$$q = 1, r = 1$$

$$0 = -2m - m^2 + 1$$

$$m = \{0.4142, -2.4142\}$$

$$g = \{0.4142, -2.4142\}$$

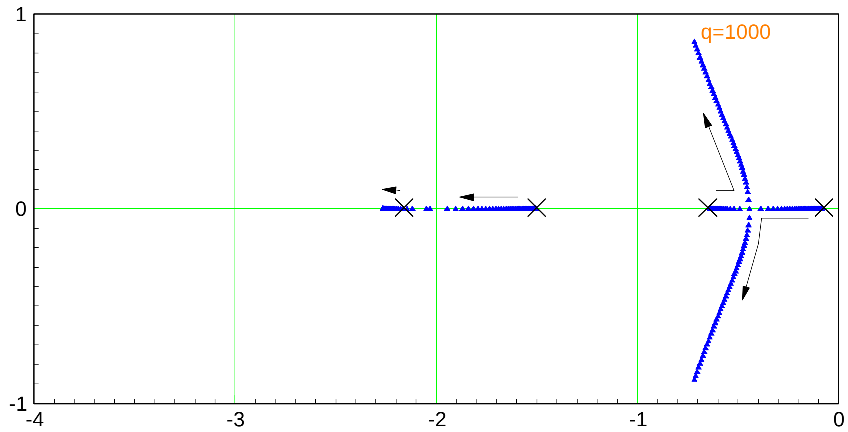
This is a typical result. m is a quadratic equation, hence there are generally two solutions. One of these solutions will be a minimum, the other a maximum. Since a feedback gain of -2.4142 results in an unstable system (A-BG has a pole at +1.4142), the second solution maximizes the cost function. The first gain, 0.4142, therefore must be the optimal feedback gain.

Choosing Q and R:

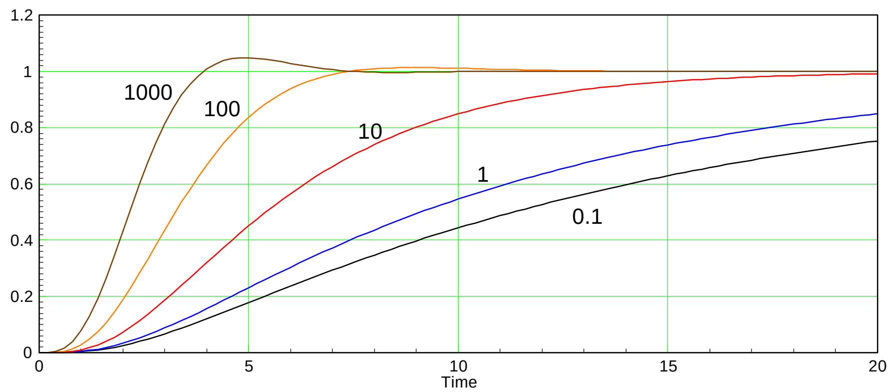
Consider the problem of designing an optimal feedback control law for the heater. If you let $R = 1$ and $Q = diag(b, 0, 0, 0)$ and plot the resulting pole locations as b varies the following results:

```
%  MATLAB Code:
%  Given A and B for the heater
data=[];
for i=1:100
   G=lqr(A,B,diag([i,0,0,0]),1);
   P=eig(A-B*G);
   data=[data;P'];
   end
plot(real(data),imag(data),'*')
```

Depending on how you choose Q and R, the optimal pole locations vary. Foe example, if Q is of the form diag([q,0,0,0]), then the optimal pole locations for 0<q<1000 are as follows:

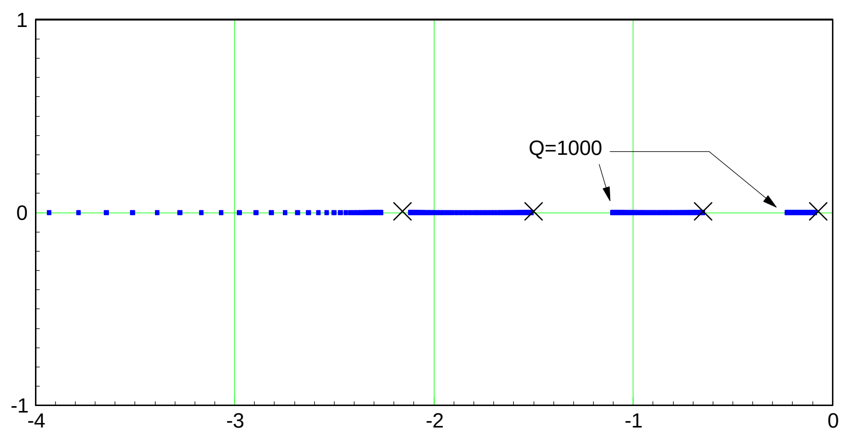


In this case, the dominant poles shift towards a damping ratio of 0.7 (which is common when Q only weights the output) and the fast poles don't shift all that much. The step response for 0.1<q<1000 follows:

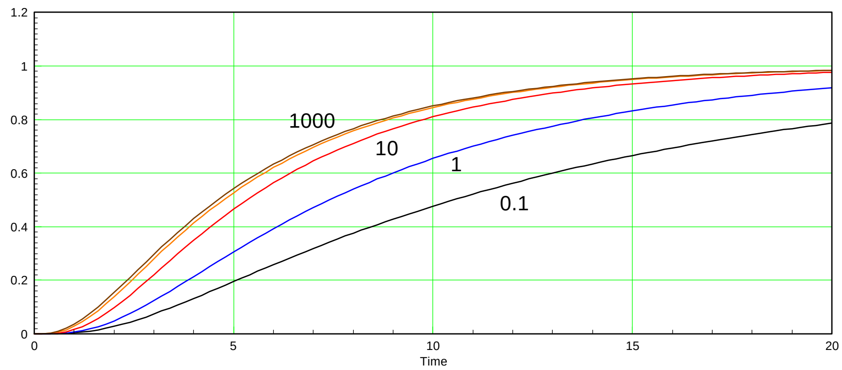


As expected, as you penalize the output more and more, it reaches steady-state faster and faster. It is also interesting that if you weight the output extremely heavily, you'll get a little overshoot in the step response (4% corresponding to a damping ratio of 45 degrees).

Alternatively, if you let Q be of the form diag([q,q,q,q]), then the optimal pole locations for 0<q<1000 are



In this case, all the poles shift left, though the dominant pole remains fairly slow. The step response is as follows:



As you weight *all* of the states more and more, youstart speeding up the system. Past Q=10, however, the system quits responding.

The explanation for this is as follows.

When you weight a state, that state tries to go to zero faster. If that state is a position (such as the system's output), the system speeds up. If that state is a velocity, then "friction" is added to the system. This results in oscillations decaying faster but real poles decaying slower.

Through a change of variable, the states *could* be position, velocity, acceleration, and jerk. By weighting all of the states, you're trying to keep the tip temperature small. You are also penalizing large changes in tip temperature (velicity) as well as large accelerations. The optimal trade-off between a fast response with small velocities and accelerations is what you see in the second set of curves.

As a result, you can use Q and R to shape the step response:

- To speed up the output, increase the weighting on the output state (or C^TC)
- To slow the system down or to reduce oscillations, increase the weighting on y' (or $(CA)^TCA$)

or a type of PD design can be developed:

$$Q = p(C^T C) + d(A^T C^T CA)$$

where p and d are the proportional and derivative "gains" used to find the full-state feedback gains.