

S-CODE: NEW MDS ARRAY CODES WITH OPTIMAL ENCODING

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ABSTRACT

In this paper, we present a new description of the X-Code, a class of MDS array code, using skew, named S-Code. The X-Codes result in codewords that are arrays of size $n \times n$, where n is prime. Our new description does not require n to be prime but requires n to be an odd number with smallest prime factor greater than 3. We prove that the S-Codes result in a distance-3 MDS code. We also give a description of which slopes other than 1 and -1 can be used to construct S-Codes.

1. X-CODE

Array codes [1] have applications in communications and storage systems [2]. Array codes use only XOR and cyclic shift operations for encoding and decoding procedures and are hence more efficient than Reed-Solomon Codes in terms of computational complexity [3].

Xu and Bruck [4] proposed a class of distance-3 MDS array codes called X-Code. The construction of X-Code is given below.

In X-Code, information symbols are placed in an array of size $(n-2) \times n$. Symbols are defined over any Abelian group with an addition operation $+$. Parity symbols are constructed from the information symbols along several parity check diagonals with the addition operation $+$. The parity symbols are placed in the bottommost two rows of the array. So the array is of size $n \times n$ where rows 0 through $n-3$ contain information symbols while rows $n-2$ and $n-1$ contain parity symbols. Each column has information symbols as well as parity symbols.

Let $C(i, j)$ be the symbol at row i and column j . The parity symbols are computed according to the following encoding rules:

$$C(n-2, i) = \sum_{k=0}^{n-3} C(k, (i+k+2) \bmod n) \quad (1)$$

$$C(n-1, i) = \sum_{k=0}^{n-3} C(k, (i-k-2) \bmod n)$$

where $i = 0, 1, \dots, n-1$. Geometrically, the two parity rows are checksums along diagonals of slopes 1 and -1 respectively. Let us see an example.

Example 1

A 5×5 X-Code array is constructed as follows. In the two schemes shown in Figures 1 and 2, every block in the array is numbered. The symbols in the blocks of the same number are added to form a parity symbol.

	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Figure 1

	0	1	2	3	4
0	0	4	3	2	1
1	1	0	4	3	2
2	2	1	0	4	3
3	3	2	1	0	4
4	4	3	2	1	0

Figure 2

Note that the last rows of both schemes are not used. An example codeword is shown in Figure 3.

	0	1	2	3	4
0	1	0	0	1	1
1	0	1	0	1	1
2	0	0	1	0	1
3	0	0	1	1	0
4	1	1	0	1	1

Figure 3

Parity symbols in row 3 correspond to row 3 of the scheme in Figure 1. Similarly, parity symbols in row 4 correspond to row 3 of the scheme in Figure 2.

The X-Codes have optimal encoding/update complexity, i.e., a change of any single data symbol affects exactly d parity symbols.

X-Code is an (n, k) code where k is the number of information rows in the codeword. A code is MDS if the code distance, d , meets the Singleton bound [5] $d \leq n - k + 1$ with equality. The X-Code is MDS because $k = n - 2$ and it is shown in [4] that the X-Code has a column distance of 3. i.e., $d = 3$. Distance-3 implies that either 2 column erasures or 1 column error can be corrected. Refer to [4] for decoding procedures of X-Codes.

2. S-CODE

In this section we use another approach to describe the construction of X-Codes. We name the code under the new construction rule S-Code. This alternative approach uses skews $(1, 1)$ and $(1, n - 1)$.

An information symbol in row i and column j of the array is referred to as $d(i, j)$, where $0 \leq i \leq n - 3$ and $0 \leq j \leq n - 1$. A parity symbol $p(n - 2, k)$ in row $(n - 2)$ where $0 \leq k \leq n - 1$ is given by the following equation:

$$p(n-2, k) = \sum_{i+j \equiv x \pmod{n}} d(i, j) \quad (2)$$

where $(n - 2) + k \equiv x \pmod{n}$. The k^{th} parity symbol in row $(n - 2)$ is the sum of the information symbols in position (i, j) such that i and j satisfy the equation $i + j \equiv x \pmod{n}$ where x is given by $(n - 2) + k \equiv x \pmod{n}$. Similarly the k^{th} parity symbol in row $(n - 1)$ is given by the following equation:

$$p(n-1, k) = \sum_{i+(n-1)j \equiv y \pmod{n}} d(i, j) \quad (3)$$

where y is given by $(n - 2) + (n - 1)k \equiv y \pmod{n}$.

In [4] it was stated that n must be a prime number. However here it is only required that n be such that the smallest prime factor of n is at least 5. i.e., n must be an odd number that does not have a prime factor of 3.

For a 5×5 S-Code array, using the above construction rule we have two schemes shown in Figures 1 and 2, respectively. In each scheme, rows 0 through 2 correspond to information symbols. One of rows 3 and 4 corresponds to the parity bits but not both. According to the construction rule, row 3 corresponds to parity bits. For illustration purposes consider the entry $(3, 1)$ in Figure 1. This entry is 4. This means that the co-ordinates (i, j) of the information symbols that will be added to compute $p(3, 1)$ are specified by the co-ordinates of the occurrences of 4's in the first three rows of Figure 1. This implies that

$$p(3, 1) = d(0, 4) + d(1, 3) + d(2, 2) \quad (4)$$

Note again that rows $n - 1$ of both schemes are not used. Row $n - 2$ of each array is respectively stored in one of the last two rows of a codeword.

3. THE MDS PROPERTY OF S-CODE

In the following lemmas and theorems we let the $(i, j)^{\text{th}}$ entry in one of the schemes resulting from (2) and (3) be $e_a(i, j)$ and that of the other scheme be $e_b(i, j)$. Note that $0 \leq i \leq n - 1$ and $0 \leq j \leq n - 1$ and the same range applies to other variables denoting a row or a column. No proofs of lemmas are given in this manuscript because of space limitations.

Lemma 1. *The scheme resulting from (2) or (3) is a Latin square of order n . i.e., no row or column contains the same entry twice.*

Lemma 2. *There do not exist (i_1, j_1) and (i_2, j_2) where $(i_1, j_1) \neq (i_2, j_2)$ such that $e_a(i_1, j_1) = e_b(i_1, j_1)$ and $e_a(i_2, j_2) = e_b(i_2, j_2)$.*

Lemma 3. *If $e_a(i_1, j_1) = e_a(i_2, j_2)$ where $(i_1, j_1) \neq (i_2, j_2)$ then there do not exist (i_1, j_1) and (i_2, j_2) such that $e_b(i_1, j_1) = e_b(n - 2, j_2)$ and $e_b(i_2, j_2) = e_b(n - 2, j_1)$.*

Lemma 4. *If $e_a(i_1, j_1) = e_a(i_2, j_2)$ and $e_a(i_3, j_2) = e_a(i_4, j_1)$ where (i_1, j_1) , (i_2, j_2) , (i_3, j_2) , and (i_4, j_1) are all distinct, then there do not exist (i_1, j_1) , (i_2, j_2) , (i_3, j_2) , and (i_4, j_1) such that $e_b(i_1, j_1) = e_b(i_3, j_2)$ and $e_b(i_2, j_2) = e_b(i_4, j_1)$.*

Theorem 1. *The S-Codes have a code distance of 3.*

Proof: Observe that S-Code is a linear code, thus proving that the code has distance 3 is equivalent to proving that a valid non-zero codeword has minimum column weight of 3, i.e., among the n columns at least 3 are non-zero. A column is non-zero if at least one symbol in it is non-zero. Let the number of non-zero columns be w . We will show that $w \geq 3$.

Suppose that in the information array, only one column, j_d ($0 \leq j_d \leq n - 1$), is non-zero. We consider two cases:

(1) Column j_d contains only one non-zero information symbol. By Lemma 1, exactly one parity symbol in row n

-2 is non-zero. Let it be in column j_{p1} ($0 \leq j_{p1} \leq n-1$) then $j_{p1} \neq j_d$. Similarly exactly one parity symbol in row $n-1$ is non-zero. Let it be in column j_{p2} ($0 \leq j_{p2} \leq n-1$) then $j_{p2} \neq j_d$. By Lemma 2, $j_{p1} \neq j_{p2}$. Therefore there exist three non-zero columns, j_d, j_{p1} , and j_{p2} , i.e., $w = 3$.

(2) Column j_d contains r ($n-2 \geq r \geq 2$) non-zero information symbols. By Lemma 1, the r non-zero information symbols do not add up to form any parity symbols. The non-zero parity symbols distribute in at least r columns. By Lemma 1 and Lemma 2, column j_d is not among these r columns. Thus $w \geq r+1$ where $r \geq 2$. i.e., $w \geq 3$.

Now suppose that in the information array, two columns, j_{d1} and j_{d2} ($0 \leq j_{d1} \leq n-1$, $0 \leq j_{d2} \leq n-1$, $j_{d1} \neq j_{d2}$), are non-zero. We consider four cases:

(1) Column j_{d1} contains only one non-zero information symbol at (i_1, j_{d1}) and column j_{d2} contains only one non-zero information symbol at (i_2, j_{d2}) .

If $e_a(i_1, j_{d1}) \neq e_a(i_2, j_{d2})$ and $e_b(i_1, j_{d1}) \neq e_b(i_2, j_{d2})$ then $w \geq 3$ due to obvious reasons.

By Lemma 2 it is impossible that $e_a(i_1, j_{d1}) = e_a(i_2, j_{d2})$ and $e_b(i_1, j_{d1}) = e_b(i_2, j_{d2})$.

If $e_a(i_1, j_{d1}) = e_a(i_2, j_{d2})$ and $e_b(i_1, j_{d1}) \neq e_b(i_2, j_{d2})$, then the array will contain exactly 2 non-zero columns if and only if $e_b(i_1, j_{d1}) = e_b(n-2, j_{d2})$ and $e_b(i_2, j_{d2}) = e_b(n-2, j_{d1})$. By Lemma 3 such (i_1, j_{d1}) and (i_2, j_{d2}) do not exist. Therefore there will be at least 3 non-zero columns. i.e., $w \geq 3$.

(2) Column j_{d1} contains only one non-zero information symbol at (i_1, j_{d1}) and column j_{d2} contains 2 non-zero information symbols at (i_2, j_{d2}) and (i_3, j_{d2}) . Then the array will contain exactly 2 non-zero columns if and only if $e_a(i_1, j_{d1}) = e_a(i_2, j_{d2})$, $e_a(i_3, j_{d2}) = e_a(n-2, j_{d1})$, $e_b(i_1, j_{d1}) = e_b(i_3, j_{d2})$, and $e_b(i_2, j_{d2}) = e_b(n-2, j_{d1})$. By Lemma 4 such (i_1, j_{d1}) , (i_2, j_{d2}) , and (i_3, j_{d2}) do not exist. Therefore there will be at least 3 non-zero columns. i.e., $w \geq 3$.

(3) Column j_{d1} contains 2 non-zero information symbols at (i_1, j_{d1}) and (i_4, j_{d1}) and column j_{d2} contains 2 non-zero information symbols at (i_2, j_{d2}) and (i_3, j_{d2}) . Then the array will contain exactly 2 non-zero columns if and only if $e_a(i_1, j_{d1}) = e_a(i_2, j_{d2})$, $e_a(i_3, j_{d2}) = e_a(i_4, j_{d1})$, $e_b(i_1, j_{d1}) = e_b(i_3, j_{d2})$, and $e_b(i_2, j_{d2}) = e_b(i_4, j_{d1})$. By Lemma 4 such (i_1, j_{d1}) , (i_2, j_{d2}) , (i_3, j_{d2}) , and (i_4, j_{d1}) do not exist. Therefore there will be at least 3 non-zero columns. i.e., $w \geq 3$.

(4) Both Columns j_{d1} and j_{d2} contain more than 2 non-zero information symbols. Since more than 2 pairs of sums will be generated among different columns, $w \geq 3$.

Lastly, suppose that in the information array three or more columns are non-zero. Then $w \geq 3$. \square

We have shown $w \geq 3$ in all cases. Therefore the S-Code has a code distance of 3.

4. S-CODE WITH SLOPES OTHER THAN 1/-1

Recall that the S-Code was constructed using skews $(1, 1)$ and $(1, n-1)$ in Section II. We now reconstruct the S-

Code using skews $(1, a)$ and $(1, b)$ such that $1 \leq a \leq n-1$ and $1 \leq b \leq n-1$. The skew (p, q) corresponds to slope q/p . i.e., we will construct the X-Code with slopes a and b .

Using skews $(1, a)$ and $(1, b)$, construction rule (2) and (3) would be modified to (5) and (6).

$$p(n-2, k) = \sum_{i+aj \equiv x \pmod{n}} d(i, j) \quad (5)$$

$$p(n-1, k) = \sum_{i+bj \equiv y \pmod{n}} d(i, j) \quad (6)$$

where $0 \leq i \leq n-3$, $0 \leq j \leq n-1$, $(n-2) + ak \equiv x \pmod{n}$, and $(n-2) + bk \equiv y \pmod{n}$.

It is worth revisiting the lemmas introduced in Section 3. Lemma 1 requires that $\gcd(a, n) = \gcd(b, n) = 1$. Lemma 2 requires that $\gcd(b-a, n) = \gcd(a-b, n) = 1$. Lemma 3 requires that $\gcd(a, b) = \gcd(2a-b, n) = \gcd(2b-a, n) = 1$, and n must not have prime factors 2 or 3. If all these conditions are satisfied then the MDS property remains. Theorem 2 follows.

Theorem 2. *If $\gcd(a, n) = \gcd(b, n) = \gcd(b-a, n) = \gcd(a-b, n) = \gcd(a, b) = \gcd(2a-b, n) = \gcd(2b-a, n) = 1$ and the smallest prime factor of n is neither 2 nor 3, then the S-Code constructed using skews $(1, a)$ and $(1, b)$ has a code distance of 3.*

Proof: This is a direct consequence of Lemmas 1 through 4 and Theorem 1. \square

Example 2

We use skews $(1, 2)$ and $(1, 3)$ to construct a 5×5 S-Code. i.e., $a = 2$ and $b = 3$, which satisfy the condition $\gcd(a, n) = \gcd(b, n) = \gcd(b-a, n) = \gcd(a-b, n) = \gcd(a, b) = \gcd(2a-b, n) = \gcd(2b-a, n) = 1$. The k^{th} ($0 \leq k \leq n-1$) entry in row 3 is given by (7):

$$p(3, k) = \sum_{i+2j \equiv x \pmod{5}} d(i, j) \quad (7)$$

where $3+2k \equiv x \pmod{5}$. Similarly the k^{th} entry in row 4 is given by (8):

$$p(4, k) = \sum_{i+3j \equiv y \pmod{5}} d(i, j) \quad (8)$$

where $3+3k \equiv y \pmod{5}$. The two resulting schemes are shown in Figures 4 and 5 respectively. An example codeword is shown in Figure 6.

5. CONCLUSION

We have presented an alternative description of the X-Code of size $n \times n$ using skews, called S-Codes. We have shown that n does not need to be a prime number. The constraint on n is that the smallest prime factor of n be at

least 5. A general condition is proposed for using other slopes to construct the S-Code. Future research will be to extend the code distance from 3 to d ($d \geq 4$). Our preliminary research shows that, if n and the skews are carefully chosen, then applying more skews (thus more parity rows) would result in larger distances.

6. ACKNOWLEDGEMENT

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7. REFERENCES

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	0	1	2	3	4
0	0	2	4	1	3
1	1	3	0	2	4
2	2	4	1	3	0
3	3	0	2	4	1
4	4	1	3	0	2

Figure 4

	0	1	2	3	4
0	0	3	1	4	2
1	1	4	2	0	3
2	2	0	3	1	4
3	3	1	4	2	0
4	4	2	0	3	1

Figure 5

	0	1	2	3	4
0	1	0	0	1	1
1	0	1	0	1	1
2	0	0	1	0	1
3	0	0	1	1	0
4	0	0	1	1	0

Figure 6